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Polynomially deformed oscillators as k-bonacci oscillators

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Abstract
A family of multi-parameter, polynomially deformed oscillators (PDOs) given by the polynomial structure function $\varphi(n)$ is studied from the viewpoint of being (or not) in the class of Fibonacci oscillators. These obey the Fibonacci relation/property (FR/FP) meaning that the nth level energy E_n is given linearly, with real coefficients, by the two preceding ones E_{n-1}, E_{n-2}. We first prove that the PDOs do not fall in the Fibonacci class. Then, three different paths of generalizing the usual FP are developed for these oscillators: we prove that the PDOs satisfy the respective k-term generalized Fibonacci (or 'k-bonacci') relations; for these same oscillators we examine two other generalizations of the FR, the inhomogeneous FR and the 'quasi-Fibonacci' relation. Extended families of deformed oscillators are studied as well: the $(q;\mu)$-oscillator with $\varphi(n)$ quadratic in the basic q-number $[n]_q$ is shown to obey the Tribonacci relation, while the $(p,q;\mu)$-oscillators with $\varphi(n)$ quadratic (cubic) in the p,q-number $[n]_{p,q}$ are proven to obey the Pentanacci (Nine-bonacci) relations. Oscillators with general $\varphi(n)$, polynomial in $[n]_q$ or $[n]_{p,q}$, are also studied.

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1. Introduction

As known for some time, among various nonlinear generalizations or deformations of the usual quantum harmonic oscillator there is a distinguished class of so-called Fibonacci oscillators [1]—the oscillators whose energy spectra satisfy the Fibonacci property (FP), implying $E_{n+1} = \lambda E_n + \rho E_{n-1}$, with real constants[2] λ and ρ. As stated in [1], the Fibonacci class is the 2-parameter deformed family of p,q-oscillators, introduced in [2]. The family

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[2] The famous Fibonacci numbers stem from the relation $F_{n+1} = F_n + F_{n-1}$ where $F_0 = F_1 = 1$.

of p, q-oscillators is rich enough. In particular, it contains such an exotic 1-parameter q-oscillator as the Tamm–Dancoff (TD) deformed oscillator [3], [4] which possesses besides the FP a set of nontrivial properties as shown in [5]. Moreover, many different 1-parameter deformed oscillators are contained in this family as particular cases. Most of them, except for the best-known q-oscillators of Arik–Cook (AC) [6] and Biedenharn–Macfarlane (BM) [7], are not well studied but nevertheless have some potential [8] for possible applications. What concerns the p, q-deformed Fibonacci oscillators, is that there exist some rather unusual properties and already elaborated interesting physical applications, see [9–16]. However, a natural question arises whether the family of p, q-oscillators exhausts the Fibonacci class. In that connection, recently we have shown in [17] that definite 3-, 4- and 5-parameter deformed extensions of the p, q-oscillator considered in [18–21] also belong to the Fibonacci class, i.e. possess the FP. In that same paper, we studied a principally different, so-called μ-deformed oscillator proposed earlier in [22], and showed that it *did not possess* the FP. For that reason, a new concept has been developed for this μ-oscillator. Namely, it was demonstrated that the μ-oscillator belongs to the more general, than Fibonacci, class of so-called quasi-Fibonacci oscillators [17].

The goal of this paper is to study yet another class of nonlinear deformed oscillators which do not belong to the Fibonacci class. We treat, from the viewpoint of three possible ways of generalizing the FP, a class of polynomially deformed oscillators. It is proven, using the notion of a deformed oscillator structure function [23–25], that those oscillators are principally of non-Fibonacci nature. Then we develop the generalization of FP for these oscillators along three completely different paths: (i) as oscillators with the k-term generalized Fibonacci property; (ii) as oscillators obeying the inhomogeneous Fibonacci relation; and (iii) as quasi-Fibonacci oscillators. Besides, we study a family of $(q; \mu)$-oscillators which is, in a sense, a mix of the quadratic and the AC type q-deformed oscillators, and demonstrate its Tribonacci property. This result is extended to a general rth order polynomial in the AC type of q-oscillator bracket $[N]_q$, naturally leading to k-bonacci relations. In this respect, let us mention that similar k-bonacci relations were treated in [26] in connection with generalized Heisenberg algebras [27]. Likewise, for the $(q; \{\mu\})$-oscillators with $\{\mu\} = (\mu_1, \mu_2, \ldots, \mu_r)$, combining the polynomial and the q-deformed AC features, the general statement on their k-bonacci property is proven. In a similar manner, the 3-parameter $(p, q; \mu)$-deformed oscillators are treated as well and shown to obey their characteristic Pentanacci property. Finally, for certain 4-parameter or $(p, q; \mu_1, \mu_2)$-deformed family of nonlinear oscillators, we demonstrate the validity of the Nine-bonacci relation by finding explicitly the relevant nine coefficients $A_j(p, q)$.

2. Polynomially deformed or $\{\mu\}$-oscillators

In the preceding work [17], we have shown that the μ-oscillator from [22] does not satisfy the usual (with *two-term* rhs) linear, homogeneous Fibonacci relation (FR)

$$E_{n+1} = \lambda E_n + \rho E_{n-1}, \tag{1}$$

with λ and ρ being some real, constant coefficients. To make the μ-deformed oscillator from [22] satisfy a relation like (1), the important modification is needed: the coefficients should depend on n: $\lambda = \lambda(n)$, $\rho = \rho(n)$ (i.e. not constants). In other words, this way of modifying the FP involves the coefficients, not the shape of relation. What concerns polynomially deformed oscillators to be studied here, we will demonstrate that they admit three different approaches for generalizing the FP.

Like in [23, 24], we study the algebra of a deformed oscillator through its *structure function*: $a^\dagger a = \varphi(N)$ and $aa^\dagger = \varphi(N+1)$. Note that the same structure function determines both the basic commutation relation of $a, a^\dagger$ and the Hamiltonian and energy eigenvalues:

$$aa^\dagger - a^\dagger a = \varphi(N+1) - \varphi(N), \tag{2}$$

$$H = \tfrac{1}{2}(\varphi(N+1) + \varphi(N)), \qquad E_n = \tfrac{1}{2}(\varphi(n) + \varphi(n+1)). \tag{3}$$

The latter formula implies usage of the properly modified version of Fock space wherein (see e.g. [25])

$$a|0\rangle = 0, \qquad N|n\rangle = n|n\rangle, \qquad \varphi(N)|n\rangle = \varphi(n)|n\rangle, \qquad H|n\rangle = E_n|n\rangle. \tag{4}$$

In this paper, we focus on the polynomially deformed oscillator. Its structure function

$$\varphi(N) = N + \sum_{i=1}^{r} \mu_i N^{i+1}, \qquad \mu_i \geqslant 0, \tag{5}$$

involves the parameters $\{\mu\} \equiv (\mu_1, \mu_2, \dots, \mu_r)$, so these polynomial oscillators may also be termed the $\{\mu\}$-deformed ones. Note that the restriction on μ_i provides positivity and monotonicity of the energies E_n in (3). It is worth remarking that for the $\{\mu\}$-deformed oscillator given by (5), the basic relation can be presented, instead of (2), also as

$$aa^\dagger - qa^\dagger a = f(N) = \sum_{l=0}^{r+1} \alpha_l N^l, \qquad \alpha_l \in \mathbf{R}. \tag{6}$$

We can translate the form (2) of basic relation into the latter one (6). Indeed, taking the q-commutator of a and $a^\dagger$ for the deformed $\{\mu\}$-oscillator, we have (we set $\mu_0 = 1$)

$$\begin{aligned} aa^\dagger - qa^\dagger a &= \varphi(N+1) - q\varphi(N) \\ &= N + 1 + \sum_{j=1}^{r} \mu_j (N+1)^{j+1} - q\Big(N + \sum_{j=1}^{r} \mu_j N^{j+1}\Big) \\ &= \sum_{j=0}^{r} \mu_j \Big(-qN^{j+1} + \sum_{s=0}^{j+1} \frac{(j+1)!}{s!(j+1-s)!} N^s\Big). \end{aligned} \tag{7}$$

The latter relation goes over into (6) if

$$\alpha_0 = \mu_0 + \mu_1 + \mu_2 + \dots + \mu_r = \sum_{s=1}^{r+1} \mu_{s-1} \tag{8}$$

$$\alpha_l = -q\mu_{l-1} + \sum_{s=1}^{r+1} \frac{s!}{l!(s-l)!} \mu_{s-1}, \qquad 1 \leqslant l \leqslant r+1. \tag{9}$$

The form of a basic relation similar to (6) was used in [28] to treat the polynomial oscillators.

2.1. *The non-Fibonacci nature of polynomial $\{\mu\}$-oscillators*

Let us first demonstrate that the polynomially deformed oscillators, see (5), do not satisfy relation (1) if one insists on the constant nature of its coefficients.

A usual quantum harmonic oscillator which has $\varphi(n) = n$ and the linear energy spectrum $E_n = \frac{1}{2}(2n+1)$ is just the particular $r = 0$ case of (5). As is known, this oscillator with $\lambda = 2$ and $\rho = -1$ satisfies the standard FR (1). Such a property, however, fails if $r = 1$, i.e. for the quadratic, with $\varphi(n) = n + \mu_1 n^2$, deformation of the harmonic oscillator cannot satisfy the standard FR (1).

The FR (1) also fails for the cubic $r = 2$ extension with $\varphi(n) = n + \mu_1 n^2 + \mu_2 n^3$ for which the energy spectrum is $E_n = \frac{1}{2}(n + \mu_1 n^2 + \mu_2 n^3 + n + 1 + \mu_1 (n+1)^2 + \mu_2 (n+1)^3)$. To show the failure, we insert the cubic $\varphi(n)$ into (1) and deduce the system of equations ($\mu_1, \mu_2 \neq 0$):

$$
\begin{aligned}
&n^3 : \mu_2 - \rho\mu_2 - \lambda\mu_2 = 0;\\
&n^2 : -\tfrac{3}{2}\lambda\mu_2 - \rho\mu_1 + \tfrac{9}{2}\mu_2 + \mu_1 + \tfrac{3}{2}\rho\mu_2 - \lambda\mu_1 = 0;\\
&n^1 : 1 + \rho\mu_1 + 3\mu_1 - \tfrac{3}{2}\lambda\mu_2 - \lambda - \lambda\mu_1 + \tfrac{15}{2}\mu_2 - \tfrac{3}{2}\rho\mu_2 - \rho = 0;\\
&n^0 : \tfrac{1}{2}\rho - \tfrac{1}{2}\rho\mu_1 + \tfrac{5}{2}\rho + \tfrac{9}{2}\mu_2 - \tfrac{1}{2}\lambda - \tfrac{1}{2}\lambda\mu_1 - \tfrac{1}{2}\lambda\mu_2 + \tfrac{1}{2}\rho\mu_2 + \tfrac{3}{2} = 0.
\end{aligned}
\tag{10}
$$

The top two equations are solved with $\lambda = 2$ and $\rho = -1$, but these values are incompatible with the rest of equations in the system, that proves the statement.

One can prove for the general situation that the rth order polynomially deformed oscillator (the structure function is of the order $r \geqslant 2$) does not satisfy the standard FR (1). Again, the equations obtained at two senior powers of n yield $\lambda = 2$ and $\rho = -1$ as a solution, but these values are incompatible with the rest of equations in the system.

Since the FP fails for the polynomial $\varphi(n)$, we consider possible extensions of the FP.

2.2. *k-term extended (k-bonacci) oscillators*

We begin with a quadratic oscillator and extend the FR by adding one term:

$$E_{n+1} = \lambda_0 E_n + \lambda_1 E_{n-1} + \lambda_2 E_{n-2}. \tag{11}$$

This is the *three-step generalized* Fibonacci or Tribonacci (see e.g. [26]) relation.

As $\lambda_0, \lambda_1, \lambda_2$ are constants, and in view of (3), it is sufficient to deal with the relation

$$\varphi_{n+1} = \lambda_0 \varphi_n + \lambda_1 \varphi_{n-1} + \lambda_2 \varphi_{n-2}, \qquad \varphi_n \equiv \varphi(n). \tag{12}$$

Indeed, if (12) is valid, relation (11) is valid too. So, insert in (12) the quadratic $\varphi(n)$ that is the $r = 1$ case of (5). Solving the system of equations deduced similarly to (10), we find $\lambda_0 = 3, \lambda_1 = -3, \lambda_2 = 1$. With these coefficients we verify that the relation (11) does hold.

Now consider the general case of polynomially deformed oscillators given by the structure function (5), with any $r \geqslant 1$. Accordingly, consider the k-term extension of FR, or the k-bonacci relation, of the form ($n \geqslant k-1$)

$$E_{n+1} = \lambda_0 E_n + \lambda_1 E_{n-1} + \lambda_2 E_{n-2} + \cdots + \lambda_{k-1} E_{n-k+1} = \sum_{i=0}^{k-1} \lambda_i E_{n-i}. \tag{13}$$

Then the following statement is true.

Proposition 1. *The energy values E_n given by (3) of the polynomially deformed oscillator with the structure function (5) satisfy the k-generalized FR (13) if $r = k-2$ and λ_i are given as*[3]

$$\lambda_i^{(k)} = (-1)^i \frac{k!}{(i+1)!(k-1-i)!} = (-1)^i \binom{k}{i+1}. \tag{14}$$

[3] Since the set of coefficients λ_i of (14) obviously depend on fixed k, we will indicate this explicitly.

Proof. Clearly, the k-term relation (13) will be valid for the energy values if the structure function given in (5) with $r = k-2$ satisfies the same equality, written as

$$n+1+\sum_{j=1}^{k-2}\mu_j(n+1)^{j+1}-\sum_{i=0}^{k-1}\lambda_i^{(k)}\left(n-i+\sum_{j=1}^{k-2}\mu_j(n-i)^{j+1}\right)=0. \tag{15}$$

The latter will be proven by induction. Supposing that the $(k-1)$-term relation

$$n+1+\sum_{j=1}^{k-3}\mu_j(n+1)^{j+1}-\sum_{i=0}^{k-2}\lambda_i^{(k-1)}\left(n-i+\sum_{j=1}^{k-3}\mu_j(n-i)^{j+1}\right)=0 \tag{16}$$

holds for the structure function $\varphi(n) = n+\sum_{i=1}^{k-3}\mu_i n^{i+1}$ with $\lambda_i^{(k-1)}$ as in (14), we then prove that the structure function $\varphi(n) = n+\sum_{i=1}^{k-2}\mu_i n^{i+1}$ satisfies relation (15) with $\lambda_i^{(k)}$ from (14). But, first let us check that (15) along with (14) is true for $k = 2, 3$.

If $k = 2$ that means the usual linear quantum oscillator, the relation is just the standard 2-term FR $E_{n+1} = \lambda_0 E_n + \lambda_1 E_{n-1}$: it does hold for $\lambda_0 = 2$ and $\lambda_1 = -1$ since for these λ_0, λ_1 the pair of relations

$$\varphi_{n+1} = \lambda_0\varphi_n + \lambda_1\varphi_{n-1}, \qquad \varphi_n = \lambda_0\varphi_{n-1} + \lambda_1\varphi_{n-2},$$

is obviously true, which read

$$n+1-2n-(-1)(n-1) = 0, \qquad n-2(n-1)-(-1)(n-2) = 0. \tag{17}$$

If $k = 3$ or for quadratic $\varphi_n = n+\mu_1 n^2$, we have the Tribonacci relation

$$\varphi_{n+1} = \lambda_0\varphi_n + \lambda_1\varphi_{n-1} + \lambda_2\varphi_{n-2}. \tag{18}$$

It rewrites in the form (15), that is

$$n+1+\mu_1(n+1)^2 = 3(n+\mu_1 n^2)+(-3)(n-1+\mu_1(n-1)^2)+1(n-2+\mu_1(n-2)^2),$$

where $\lambda_0 = 3$, $\lambda_1 = -3$ and $\lambda_2 = 1$. The latter relation, with account of both the identities in (17), reduces to

$$\mu_1(n+1)^2 - 3\mu_1 n^2 + 3\mu_1(n-1)^2 - \mu_1(n-2)^2 = 0,$$

where we encounter the full squares only. This, as easily checked, holds identically.

Similar reasonings are applied to the situation of a general polynomial $\varphi(n)$. Note first that $\lambda_i^{(k)}$ in (14) split as

$$\lambda_i^{(k)} = \lambda_i^{(k-1)} - \lambda_{i-1}^{(k-1)}. \tag{19}$$

Using this splitting on the LHS of the kth order generalized Fibonacci (k-bonacci) relation (15), we extract *twice* the (supposed to hold) $(k-1)$-term generalized FR (16): first, in the form of lhs of (16), for fixed n, with $\lambda_i^{(k-1)}$ involved and, second, in the form of LHS of (16) rewritten for $n \to n-1$ and involving the set $(-1)\lambda_{i-1}^{(k-1)}$ from (19). As a result, we get the relation consisting of the highest $(k-1)$st order terms (in $n+1$ or in $n-i$) only:

$$\mu_{k-1}(n+1)^{k-1} - \sum_{i=0}^{k-1}\lambda_i^{(k)}(\mu_{k-1}(n-i)^{k-1}) = 0. \tag{20}$$

Since $\mu_{k-1} \neq 0$, the latter relation is rewritten as

$$F_n(k, \lambda_i^{(k)}) \equiv (n+1)^{k-1} - \sum_{i=0}^{k-1} \lambda_i^{(k)} (n-i)^{k-1} = 0. \tag{21}$$

Then, to prove (21), we expand the binomials and interchange the summation order:

$$\begin{aligned}
F_n(k, \lambda_i^{(k)}) = &\sum_{s=0}^{k-1} \frac{(k-1)!}{s!(k-1-s)!} n^{k-1-s} 1^s \\
&- \sum_{i=0}^{k-1} (-1)^i \frac{k!}{(i+1)!(k-1-i)!} \sum_{s=0}^{k-1} \frac{(k-1)!}{s!(k-1-s)!} n^{k-1-s} (-1)^s i^s \\
= &\sum_{s=0}^{k-1} \frac{(k-1)!}{s!(k-1-s)!} n^{k-1-s} \left(1 - (-1)^s k! \sum_{i=0}^{k-1} (-1)^i \frac{i^s}{(i+1)!(k-1-i)!} \right) \\
= &\sum_{s=0}^{k-1} \frac{(k-1)!}{s!(k-1-s)!} n^{k-1-s} \left((-1)^{s+1} \sum_{-1 \leqslant i \leqslant k-1} (-1)^i \frac{k!}{(i+1)!(k-1-i)!} i^s \right)
\end{aligned}$$

(note that the entity 1 is included in the sum as the additional $i = -1$ term).

Shifting the index i as $i \to i-1$ we obtain

$$\begin{aligned}
F_n(k, \lambda_i^{(k)}) = &\sum_{s=0}^{k-1} \frac{(k-1)!}{s!(k-1-s)!} n^{k-1-s} \left((-1)^{s+1} \sum_{-1 \leqslant i-1 \leqslant k-1} (-1)^{i-1} \frac{k!}{i!(k-i)!} (i-1)^s \right) \\
= &\sum_{i=0}^{k-1} \frac{(k-1)!}{s!(k-1-s)!} n^{k-1-s} \left((-1)^s \sum_{i=0}^{k} (-1)^i \frac{k!}{i!(k-i)!} (i-1)^s \right) = 0
\end{aligned}$$

where the fact of final turning into zero is due to the formula

$$\sum_{j=0}^{k} (-1)^j \frac{k!}{j!(k-j)!} (j-1)^m = 0, \qquad m = 0, 1, 2, \ldots, k-1,$$

which can be proven analogously to the known formula [29]

$$\sum_{j=0}^{k} (-1)^j \frac{k!}{j!(k-j)!} j^m = 0, \qquad m = 0, 1, 2, \ldots, k-1.$$

Thus we gain the proof. □

Remark 1. It is remarkable that the set (14) of the coefficients $\lambda_i^{(k)}$, $i = 0, 1, 2, \ldots, k-1$, which provide the validity of the k-term Fibonacci relation (13) for the polynomially deformed oscillators with the structure function $\varphi(n) = n + \sum_{i=1}^{k-2} \mu_i n^{i+1}$, see (5), are *totally independent* of the parameters μ_i of $\varphi(n)$. In particular, some of the μ_i (but not the 'senior' one μ_{k-2}) may be equal to zero.

Remark 2. The content of proposition 1 can be extended to the case $r < k-2$ or $r > k-2$. Namely, it can be demonstrated that the k-term Fibonacci relation is satisfied for all the polynomial oscillators for which $r < k-2$. Equivalently, the oscillator with rth order polynomial structure function satisfies all the k-term generalized FR such that $k > r+2$, with appropriate coefficients. For instance, the quadratic oscillator which obeys the 3-term or

Tribonacci relation, see (18), with fixed λ_0, λ_1 and λ_2 equal respectively to 3, −3, 1, obviously satisfies, with definite four coefficients, also the 4-term relation

$$\varphi_{n+1} = (\lambda_0 - 1)\varphi_n + (\lambda_0 + \lambda_1)\varphi_{n-1} + (\lambda_1 + \lambda_2)\varphi_{n-2} + \lambda_2\varphi_{n-3},$$

and with proper coefficients also the higher order 5-term, 6-term, etc k-bonacci relations. On the other hand, the oscillator with *the* rth order polynomial structure function does not satisfy any k-term generalized Fibonacci relations such that $k < r + 2$. Accordingly, the k-term Fibonacci relation is not valid for those polynomial oscillators for which $r > k - 2$.

2.3. Polynomial $\{\mu\}$-oscillators: inhomogeneous FR

Here we consider an alternative (though also linear in the energy eigenvalues) form of generalized FR which is valid for the polynomially deformed oscillators:

$$E_{n+1} = \lambda E_n + \rho E_{n-1} + \sum_{i=0}^{k-1} \alpha_i n^i, \qquad E_n = \frac{1}{2}(\varphi(n) + \varphi(n+1)). \qquad (22)$$

For obvious reasons and in analogy with [30], we call such an extension of FR the 'inhomogeneous Fibonacci relation'. Again it is sufficient to deal, instead of the energy by itself, with the structure function (5) of the deformed oscillator. So, consider two relations

$$\varphi(n+1) = \lambda\varphi(n) + \rho\,\varphi(n-1) + \sum_{i=0}^{k-1} \tilde{\alpha}_i\, n^i,$$

$$\varphi(n+2) = \lambda\varphi(n+1) + \rho\,\varphi(n) + \sum_{i=0}^{k-1} \tilde{\tilde{\alpha}}_i\, n^i. \qquad (23)$$

The validity of these two equations will guarantee fulfillment of the inhomogeneous Fibonacci relation (22) if in addition we require $\tilde{\alpha}_i + \tilde{\tilde{\alpha}}_i = \alpha_i$.

It can be shown that the $(k+2)$-term[4] inhomogeneous FR is satisfied for all the polynomial oscillators for which $r = k$. Inversely, the oscillator with $(r+1)$st order polynomial structure function satisfies any $(k+2)$-term inhomogeneous FR such that $k > r$. On the other hand, the oscillator with the rth order polynomial structure function does not satisfy all the k-term generalized inhomogeneous FRs such that $k < r$. Accordingly, the k-term FR is not valid for all the polynomial oscillators for which $r > k$.

Instead of proving the general statements of the latter paragraph we only give particular examples, the necessary data for which are given in table 1.

The quadratically deformed oscillator, with $r = 1$ or $\varphi(n) = n + \mu_1 n^2$, does not satisfy the standard FR (1), but it obeys the simplest inhomogeneous FR:

$$E_{n+1} = \lambda E_n + \rho E_{n-1} + \alpha_0, \qquad \lambda = 2, \quad \rho = -1, \quad \alpha_0 = 4\mu_1; \qquad (24)$$

see the $k = 1$ row in the table. Let us note that, whatever is k (i.e. for any power in n of the polynomial structure function), the coefficients λ, ρ will always be $\lambda = 2$, $\rho = -1$. The set $\alpha_0, \alpha_1, \ldots$, however, differs for different k, as seen in the five rows of the table.

Remark that, contrary to the case of the k-bonacci relation where all the coefficients $\lambda_i^{(k)}$ in (14) are really constant (independent of *n and* $\{\mu\}$), here α_i are functions of μ_j.

[4] We count all the terms in (22), including the λ-term and ρ-term.

Table 1. The coefficients α_i, $\tilde{\alpha}_i$ and $\tilde{\tilde{\alpha}}_i$ in (22)–(23) for a few low values of k.

	Coefficients $\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k$ from (23)	Coefficients $\tilde{\tilde{\alpha}}_0, \tilde{\tilde{\alpha}}_1, \ldots, \tilde{\tilde{\alpha}}_k$ from (23)	Coefficients $\alpha_0, \alpha_1, \ldots, \alpha_k$ from (22)
$k = 1$	$\tilde{\alpha}_0 = 2\mu_1$	$\tilde{\tilde{\alpha}}_0 = 2\mu_1$	$\alpha_0 = 4\mu_1$
$k = 2$	$\tilde{\alpha}_0 = 2\mu_1$	$\tilde{\tilde{\alpha}}_0 = 2\mu_1 + 6\mu_2$	$\alpha_0 = 4\mu_1 + 6\mu_2$
	$\tilde{\alpha}_1 = 6\mu_2$	$\tilde{\tilde{\alpha}}_1 = 6\mu_2$	$\alpha_1 = 12\mu_2$
$k = 3$	$\tilde{\alpha}_0 = 2\mu_1 + 2\mu_3$	$\tilde{\tilde{\alpha}}_0 = 2\mu_1 + 6\mu_2 + 14\mu_3$	$\alpha_0 = 4\mu_1 + 6\mu_2 + 16\mu_3$
	$\tilde{\alpha}_1 = 6\mu_2$	$\tilde{\tilde{\alpha}}_1 = 6\mu_2 + 24\mu_3$	$\alpha_1 = 12\mu_2 + 24\mu_3$
	$\tilde{\alpha}_2 = 12\mu_3$	$\tilde{\tilde{\alpha}}_2 = 12\mu_3$	$\alpha_2 = 24\mu_3$
$k = 4$	$\tilde{\alpha}_0 = 2\mu_1 + 2\mu_3$	$\tilde{\tilde{\alpha}}_0 = 2\mu_1 + 6\mu_2$	$\alpha_0 = 4\mu_1 + 6\mu_2$
		$+ 14\mu_3 + 30\mu_4$	$+ 16\mu_3 + 30\mu_4$
	$\tilde{\alpha}_1 = 6\mu_2 + 10\mu_4$	$\tilde{\tilde{\alpha}}_1 = 6\mu_2 + 24\mu_3 + 70\mu_4$	$\alpha_1 = 12\mu_2 + 24\mu_3 + 80\mu_4$
	$\tilde{\alpha}_2 = 12\mu_3$	$\tilde{\tilde{\alpha}}_2 = 12\mu_3 + 60\mu_4$	$\alpha_2 = 24\mu_3 + 60\mu_4$
	$\tilde{\alpha}_3 = 20\mu_4$	$\tilde{\tilde{\alpha}}_2 = 20\mu_4$	$\alpha_2 = 40\mu_4$
$k = 5$	$\tilde{\alpha}_0 = 2\mu_1 + 2\mu_3 + 2\mu_5$	$\tilde{\tilde{\alpha}}_0 = 2\mu_1 + 6\mu_2 + 14\mu_3$	$\alpha_0 = 4\mu_1 + 6\mu_2 + 16\mu_3$
		$+ 30\mu_4 + 62\mu_5$	$+ 30\mu_4 + 64\mu_5$
	$\tilde{\alpha}_1 = 6\mu_2 + 10\mu_4$	$\tilde{\tilde{\alpha}}_1 = 6\mu_2 + 24\mu_3$	$\alpha_1 = 12\mu_2 + 24\mu_3$
		$+ 70\mu_4 + 180\mu_5$	$+ 80\mu_4 + 180\mu_5$
	$\tilde{\alpha}_2 = 12\mu_3 + 30\mu_5$	$\tilde{\tilde{\alpha}}_2 = 12\mu_3 + 60\mu_4 + 210\mu_5$	$\alpha_2 = 24\mu_3 + 60\mu_4 + 240\mu_5$
	$\tilde{\alpha}_3 = 20\mu_4$	$\tilde{\tilde{\alpha}}_3 = 20\mu_4 + 120\mu_5$	$\alpha_3 = 40\mu_4 + 120\mu_5$
	$\tilde{\alpha}_4 = 30\mu_5$	$\tilde{\tilde{\alpha}}_4 = 30\mu_5$	$\alpha_4 = 60\mu_5$

2.4. Polynomially deformed oscillators as quasi-Fibonacci ones

In subsection 2.2, we assumed the coefficients λ_i, $i = 0, \ldots, k-1$, in the k-generalized Fibonacci (or k-bonacci) relation (13) to be real constants. In this subsection we modify the initial two-term linear, *standard* FR (1) by admitting an explicit dependence on the number n of both λ and ρ entering the relation. That is, now we deal with the so-called *quasi-Fibonacci relation*[5]:

$$E_{n+1} = \lambda(n)E_n + \rho(n)E_{n-1}. \tag{25}$$

Let us note that for the μ-oscillator from [22], which is non-Fibonacci, its quasi-Fibonacci properties have been described in detail in [17] where three different ways of deriving λ_n and ρ_n have been explored. Here, for the polynomially deformed or $\{\mu\}$-oscillators, only two of the three are considered.

Following the first way, we deal with the system of equations related to (25), namely

$$\begin{cases} \varphi(n+1) = \lambda_n\varphi(n) + \rho_n\varphi(n-1); \\ \varphi(n+2) = \lambda_n\varphi(n+1) + \rho_n\varphi(n). \end{cases} \tag{26}$$

The simultaneous validity of both of them guarantees fulfillment of (25). Solving (26) yields

$$\lambda_n = \frac{\varphi(n+1) - \rho_n\,\varphi(n-1)}{\varphi(n)}, \qquad \rho_n = \frac{\varphi(n+2)\varphi(n) - \varphi^2(n+1)}{\varphi^2(n) - \varphi(n+1)\varphi(n-1)}. \tag{27}$$

[5] Below, for convenience, we denote $\lambda(n)$ and $\rho(n)$ also as λ_n and ρ_n.

With account of the explicit form (5) of the structure function, we have

$$\rho_n = \frac{\sum_{i=0}^{k}\mu_i n^{i+1}\sum_{j=0}^{k}\mu_j(n+2)^{j+1} - \sum_{i=0}^{k}\mu_i(n+1)^{i+1}\sum_{j=0}^{k}\mu_j(n+1)^{j+1}}{\sum_{i=0}^{k}\mu_i n^{i+1}\sum_{j=0}^{k}\mu_j n^{j+1} - \sum_{i=0}^{k}\mu_i(n-1)^{i+1}\sum_{j=0}^{k}\mu_j(n+1)^{j+1}}.$$

The obtained expression for ρ_n by plugging it in equation (27) yields also λ_n.

To proceed in the second way, see [17], we put $\rho_n = \lambda_{n-1}$ in (12), which gives

$$E_{n+1} = \lambda_n E_n + \lambda_{n-1}E_{n-1}$$

or

$$\lambda_{n+1} + \frac{E_n}{E_{n+1}}\lambda_n = \frac{E_{n+2}}{E_{n+1}}, \qquad n \geqslant 0. \tag{28}$$

With the initial condition $\lambda_0 = c$, we find by induction the formula

$$\lambda_n \equiv \lambda(n) = \frac{\sum_{j=2}^{n+1}(-1)^{n-j+1}E_j + (-1)^n cE_0}{E_n},$$

which in terms of the structure function looks as

$$\lambda_n = \frac{\sum_{j=2}^{n+1}(-1)^{n-j+1}\varphi(j) + (-1)^n c\varphi(0)}{\varphi(n)}.$$

With account of (5), the expressions for λ_n and $\rho_n = \lambda_{n-1}$ which provide validity of the quasi-Fibonacci relation (25) take the final explicit form

$$\lambda_n = \frac{\sum_{j=2}^{n+1}(-1)^{n-j+1}\sum_{i=0}^{s}\mu_i j^{i+1}}{\sum_{i=0}^{s}\mu_i n^{i+1}}, \qquad \rho_n = \frac{\sum_{j=2}^{n}(-1)^{n-j}\sum_{i=0}^{s}\mu_i k^{i+1}}{\sum_{i=0}^{s}\mu_i(n-1)^{i+1}}. \tag{29}$$

This completes our short quasi-Fibonacci treatment of the polynomially deformed $\{\mu\}$-oscillators, $\mu \equiv (\mu_1, \mu_2, \ldots, \mu_n)$.

3. Deformed oscillators, polynomial in q- or p, q-brackets

Here we examine some other, than the $\{\mu\}$-deformed, classes of oscillators (with added more canonical deformation parameters) obeying Tribonacci and higher order relations.

3.1. A class of $(q;\mu)$- and $(q;\{\mu\})$-deformed oscillators

Consider the $(q;\mu)$-deformed oscillator defined by the structure function

$$\varphi_n(q;\mu) \equiv \varphi(n;q,\mu) = [n] + \mu[n]^2 = [n]\big(1+\mu[n]\big), \tag{30}$$

$$[n] \equiv [n]_q = \frac{1-q^n}{1-q}, \qquad q > 0. \tag{31}$$

It can be proven that the structure function (30) and hence the energy values E_n of such oscillators obey the 3-term extended Fibonacci (=Tribonacci) relation

$$\varphi_{n+1}(q,\mu) = \lambda(q)\,\varphi_n(q,\mu) + \rho(q)\,\varphi_{n-1}(q,\mu) + \sigma(q)\,\varphi_{n-2}(q,\mu), \tag{32}$$

$$E_{n+1} = \lambda(q)E_n + \rho(q)E_{n-1} + \sigma(q)E_{n-2}, \tag{33}$$

where $\lambda(q)$, $\rho(q)$, $\sigma(q)$ depend on the parameter q as

$$\lambda(q) = [3], \qquad \rho(q) = -q[3], \qquad \sigma(q) = q^3$$

(compare with (12) and its coefficients $\lambda_0 = 3, \lambda_1 = -3, \lambda_2 = 1$). The result in (30), (32)–(33), with the above $\lambda(q)$, $\rho(q)$, $\sigma(q)$, generalizes to the following statement.

Proposition 2. *For the* $(q; \{\mu\})$*-deformed oscillators, their structure function*

$$\varphi_n(q; \{\mu\}) = [n]_q + \sum_{j=1}^{r} \mu_j\, ([n]_q)^{j+1}, \qquad \{\mu\} \equiv (\mu_1, \mu_2, \ldots, \mu_r), \tag{34}$$

and thus the energies E_n *obey the k-term extended Fibonacci (or 'k-bonacci') relation*

$$\varphi_{n+1}(q; \{\mu\}) = \lambda_0\, \varphi_n(q; \{\mu\}) + \lambda_1\, \varphi_{n-1}(q; \{\mu\}) + \cdots + \lambda_{k-1}\, \varphi_{n-k+1}(q; \{\mu\}), \tag{35}$$

$$E_{n+1} = \lambda_0\, E_n(q; \{\mu\}) + \lambda_1\, E_{n-1}(q; \{\mu\}) + \cdots + \lambda_{k-1}\, E_{n-k+1}(q; \{\mu\}), \tag{36}$$

if $r = k - 2$ *and the coefficients* $\lambda_i = \lambda_i(q)$, $i = 0, 1, \ldots, k-1$, *are taken in the form*

$$\lambda_i(q) = \lambda_i^{(k)}(q) = (-1)^i\, q^{i(i+1)/2} \frac{[k]!}{[i+1]![k-1-i]!} = (-1)^i\, q^{i(i+1)/2} \begin{pmatrix} k \\ i+1 \end{pmatrix}_q . \tag{37}$$

The proof proceeds in analogy with that of proposition 1.

Note: in the limit $q \to 1$, formulas (34) and (37) reduce respectively to (5) and (14), that in this limit gives complete recovery of proposition 1. As an interesting fact let us stress the independence of $\lambda_i^{(k)}(q)$ on $\{\mu\}$ in (37), and in this limit.

3.2. A class of $(p, q; \mu)$-deformed nonlinear oscillators

Let us recall that the q-deformed oscillator linear in the q-bracket (31) possesses, on one hand, the standard 2-term Fibonacci property (1) while, on the other hand, is a particular $p = 1$ case of the p, q-deformed oscillator whose p, q-bracket is

$$[n]_{p,q} \equiv \frac{p^n - q^n}{p - q}. \tag{38}$$

We could expect that the 3-term (Tribonacci) relation holds for the oscillators involving besides μ two more deformation parameters p, q, so that the structure function is

$$\varphi_n(p, q; \mu) = [n]_{p,q} + \mu[n]_{p,q}^2 = [n]_{p,q}\big(1 + \mu[n]_{p,q}\big). \tag{39}$$

But, it turns out that this fails. To prove this fact, suppose the opposite that deformed oscillator with the structure function (39) obeys the Tribonacci relation

$$\varphi_{n+1}(p, q; \mu) = \lambda(p, q)\varphi_n(p, q; \mu) + \rho(p, q)\varphi_{n-1}(p, q; \mu) + \sigma(p, q)\varphi_{n-2}(p, q; \mu). \tag{40}$$

Insert (39) into relation (40) and by equating the corresponding coefficients deduce the following set of equations:

$$\begin{aligned}
&p^n : p^2 - pq = \lambda(p - q) + \rho(1 - p^{-1}q) + \sigma(p^{-1} - p^{-2}q),\\
&q^n : q^2 - pq = \lambda(q - p) + \rho(1 - q^{-1}p) + \sigma(q^{-1} - q^{-2}p),\\
&(pq)^n : pq = \lambda + \rho p^{-1}q^{-1} + \sigma p^{-2}q^{-2},\\
&p^{2n} : p^2 = \lambda + \rho p^{-2} + \sigma p^{-4},\\
&q^{2n} : q^2 = \lambda + \rho q^{-2} + \sigma q^{-4}.
\end{aligned}$$

This system of equations is easily shown to be inconsistent (having no solutions). Therefore, the structure function (39) does not satisfy the Tribonacci relation (40).

Analogously to this negative result, it can be proven that the deformed oscillator under question does not satisfy as well the 4-term (or Tetranacci) relation.

In the next subsection we will show how to properly treat the deformed oscillators defined by the structure function (39) and alike, from the viewpoint of yet higher extension of the Fibonacci (Tribonacci, Tetranacci) relations.

3.3. Deformed $(p, q; \mu)$-oscillators as Pentanacci oscillators

One can prove that the following statement is true.

Proposition 3. *The family of $(p, q; \mu)$-oscillators with quadratic in $[n]_{p,q}$ structure function (39) obeys the Pentanacci (5-term extended Fibonacci) relation*

$$\varphi_{n+1} = \lambda(p, q)\varphi_n + \rho(p, q)\varphi_{n-1} + \sigma(p, q)\varphi_{n-2} + \gamma(p, q)\varphi_{n-3} + \delta(p, q)\varphi_{n-4} \tag{41}$$

if the coefficients $\lambda(p, q), \rho(p, q), \sigma(p, q), \gamma(p, q), \delta(p, q)$ are[6]

$$\begin{aligned}
\lambda(p, q) &= p^2 + q^2 + p + q + pq = [2]_{p,q} + [3]_{p,q},\\
\rho(p, q) &= -p^3q - p^3 - 2p^2q - p^2q^2 - pq^3 - 2pq^2 - pq - q^3\\
&= -([4]_{p,q} + pq(1 + [2]_{p,q} + [3]_{p,q})) = -([3]_{p,q} \cdot [2]_{p,q} + pq(1 + [3]_{p,q})),\\
\sigma(p, q) &= 2p^3q^2 + 2p^2q^3 + p^4q + pq^4 + p^3q + pq^3 + p^2q^2 + p^3q^3\\
&= pq([3]_{p,q}([2]_{p,q} + 1) + p^2q^2),\\
\gamma(p, q) &= -(p^2q + p^2 + pq^2 + pq + q^2)p^2q^2 = -p^2q^2([3]_{p,q} + pq[2]_{p,q}),\\
\delta(p, q) &= p^4q^4.
\end{aligned} \tag{42}$$

The proof proceeds by direct verification.

Remark 3. One can show that these same oscillators satisfy also the respective k-term-extended Fibonacci relation for an integer $k \geqslant 5$. Let us illustrate this for $k = 6$. We take one more copy of relation (41) in which the shift $n \to n - 1$ is done, and subtract this copy, multiplied by some κ, from the initial relation (41). Then the 6-term extended relation

$$\varphi_{n+1} = (\lambda - \kappa)\varphi_n + (\rho + \lambda\kappa)\varphi_{n-1} + (\sigma + \rho\kappa)\varphi_{n-2} + (\gamma + \sigma\kappa)\varphi_{n-3} + (\delta + \gamma\kappa)\varphi_{n-4} + \kappa\varphi_{n-5}$$

results and is valid for any real number κ, with the coefficients $\lambda, \rho, \sigma, \gamma, \delta$ taken from (42). Note that this same procedure can be applied any desired number of times, with the appropriate shifts $n \to n - j$ (clearly, $j < n$).

Remark 4. It is of interest to check the $p = 1$ limit of (39) and (41)–(42). Contrary to naive expectation that we should obtain the $r = 5$ case of proposition 2, with the coefficients $\lambda_i^{(5)}$, given in (37), we find a kind of surprise: the coefficients that follow from (42) are other than $\lambda_0^{(5)}, \lambda_1^{(5)}, \lambda_2^{(5)}, \lambda_3^{(5)}$ and $\lambda_4^{(5)}$. This 'controversy' is rooted in the fact that the $(q; \mu)$-oscillator with $\varphi(n) = [n]_q + \mu[n]_q^2$ already respects the Tribonacci (3-term extended Fibonacci) relation while usual FR fails. The situation with the $(p, q; \mu)$-oscillator is more involved: for it, both the usual 2-term FR and the 3-, 4-term extensions of Fibonacci relations are not valid; only the 5-term or Pentanacci relation does hold.

[6] Note their μ-independence.

3.4. Nine-bonacci deformed $(p, q; \mu_1, \mu_2)$-oscillators

Consider the cubic in the p, q-bracket $[n]_{p,q}$ structure function:

$$\varphi_n = [n]_{p,q} + \mu_1[n]^2_{p,q} + \mu_2[n]^3_{p,q}, \qquad [n]_{p,q} = \frac{p^n - q^n}{p - q}. \tag{43}$$

It can be proven that such an oscillator does not satisfy the standard FR, nor it satisfies any k-term extended, $k \leqslant 8$, k-bonacci relation. However, it does satisfy the 9-term extended version of FR (*'Nine-bonacci relation'*), as reflected in the next statement.

Proposition 4. *The $(p, q; \mu_1, \mu_2)$-oscillator given by the structure function $\varphi(n)$ in (43) satisfies the 9-term extension of the FR or Nine-bonacci relation of the form*

$$\varphi_{n+1} = \sum_{j=0}^{8} A_j\, \varphi_{n-j} \tag{44}$$

if the coefficients $A_j \equiv A_j(p, q)$ are given as (note their μ_1, μ_2-independence)

$$\begin{aligned}
A_0(p,q) &= [4]_{p,q} + [3]_{p,q} + [2]_{p,q},\\
A_1(p,q) &= -([6]_{p,q} + (1+pq)[5]_{p,q} + (1+pq)[4]_{p,q} + 2pq[3]_{p,q}\\
&\quad + pq(1+pq)[2]_{p,q} + pq(1+p^2q^2)),\\
A_2(p,q) &= (1+pq)[7]_{p,q} + 2pq[6]_{p,q} + pq(2+pq)[5]_{p,q} + pq(2+4pq+p^2q^2)[4]_{p,q}\\
&\quad + pq(1+2pq+2p^2q^2)[3]_{p,q} + pq(pq+2p^2q^2)[2]_{p,q} + 2p^3q^3,\\
A_3(p,q) &= -pq([8]_{p,q} + (1+pq)[7]_{p,q} + (1+2pq+p^2q^2)[6]_{p,q} + pq(3+2pq)[5]_{p,q}\\
&\quad + pq(1+4pq+p^2q^2)[4]_{p,q} + pq(1+2pq+3p^2q^2)[3]_{p,q}\\
&\quad + p^2q^2(2+2pq+p^2q^2)[2]_{p,q} + p^3q^3(2+p^2q^2)),\\
A_4(p,q) &= p^2q^2([8]_{p,q} + (1+pq)[7]_{p,q} + (1+2pq+p^2q^2)[6]_{p,q})\\
&\quad + p^3q^3((2+3pq)[5]_{p,q} + (1+4pq+p^2q^2)[4]_{p,q})\\
&\quad + p^4q^4((3+2pq+p^2q^2)[3]_{p,q} + (1+2pq+2p^2q^2)[2]_{p,q} + (1+2p^2q^2)),\\
A_5(p,q) &= -p^3q^3((1+pq)[7]_{p,q} + 2pq[6]_{p,q} + pq(1+2pq)[5]_{p,q}\\
&\quad + pq(1+2pq+2p^2q^2)[4]_{p,q} + p^2q^2(2+2pq+p^2q^2)[3]_{p,q}\\
&\quad + p^3q^3(2+pq)[2]_{p,q} + 2p^4q^4),\\
A_6(p,q) &= p^7q^7([6]_{p,q} + 2[3]_{p,q} + (1+pq)[2]_{p,q} + (2+p^2q^2))\\
&\quad + p^5q^5(1+pq)[5]_{p,q} + p^6q^6(1+pq)[4]_{p,q},\\
A_7(p,q) &= -p^7q^7([4]_{p,q} + pq[3]_{p,q} + p^2q^2[2]_{p,q}),\\
A_8(p,q) &= p^{10}q^{10}.
\end{aligned} \tag{45}$$

The proof is achieved by direct verification.

Remark 5. If $p \to 1$, (43) reduces to the $r = 2$ case of (34), and the coefficients (45) turn into

$$\begin{aligned}
A_0(q) &= [4]_q + [3]_q + [2]_q,\\
A_1(q) &= -2q^2([4]_q + [3]_q + 2q) - q^2[2]_q,\\
A_2(q) &= [8]_q + 5q[6]_q + 2q[5]_q + q[4]_q + 6q^2[3]_q + q^3[2]_q + 6q^3,\\
A_3(q) &= -q(3[8]_q + 6q[6]_q + 2q[5]_q + 5q^2[4]_q + 6q^3[3]_q + 8q^3[2]_q + 2q^4),\\
A_4(q) &= 3q^2[8]_q + 6q^3[6]_q + 2q^4[5]_q + 11q^4[4]_q + 4q^5[2]_q + 4q^6,\\
A_5(q) &= -q^3([8]_q + 6q[6]_q + q[5]_q + 4q^2[4]_q + 3q^3[3]_q + 3q^3[2]_q + 4q^4,\\
A_6(q) &= 2q^5[6]_q + q^6[5]_q + q^6[4]_q + 2q^7[3]_q + 3q^8[2]_q + 3q^9,\\
A_7(q) &= -q^7([4]_q + q[3]_q + q^2[2]_q),\\
A_8(q) &= q^{10}. \qquad (46)
\end{aligned}$$

Here again we find a surprise: the $p \to 1$ limits of the coefficients A_j do not merge with those stemming from the general formula (37). This is rooted in the fact that the structure function (43) goes over into $\varphi(n) = [n]_q + \mu_1[n]_q^2 + \mu_2[n]_q^3$, for which the Pentanacci relation is already valid as follows from proposition 2. To lift the controversy, let us show that it is possible to derive the 9-term extended FR (44), just with the coefficients (46), starting from the Pentanacci relation given by (41)–(42) at $k = 4$. Indeed, consider besides the Pentanacci relation (41) at $k = 4$, with fixed n, the four additional copies of it written accordingly other shifts $n \to n-1, n \to n-2, n \to n-3, n \to n-4$. Then multiply these four relations respectively by t, x, y, z of the form

$$\begin{aligned}
t &= -([2] + [3]),\\
x &= -(2q^2 + 1)([4] + [3]) - [2](q^2 - 1) - 4q^3,\\
y &= -[8] - 5q[6] - 2q[5] + [4](2q^2 - q + 2) + [3](-4q^2 + 2) + [2](-q^3 + q^2 - 1) - 2q^3,\\
z &= 3q[8] + [6](6q^2 + 5q) + 2q[5](q + 1) + [4]q(5q^2 + 1) + [3](6q^4 + 6q^2 + 4)\\
&\quad + [2](8q^4 + q^3 + 1) + 2q^5 + 6q^3,
\end{aligned}$$

and take their sum, term by term, with the first copy. That will lead to the above Nine-bonacci relation (44), with exactly the coefficients given in (46).

We conclude this section with the *remark concerning general situation* of deformed oscillators with a polynomial of any order in the $[n]_{p,q}$ structure function, cf (43). It can be argued that the oscillator for which $\varphi(n)$ is the $(k+1)$st order polynomial in $[n]_{p,q}$ obeys the m-bonacci relation where $m = \frac{1}{2}(k+2)(k+3) - 1$. It is, however, hardly possible to find the coefficients $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ in the explicit form *for arbitrary m*.

4. Conclusions

In this paper we have shown, first, that deformed oscillators whose structure function is a polynomial in n, or $[n]_q$, or $[n]_{p,q}$, do not belong to the Fibonacci class. On the other hand, the $\{\mu\}$-oscillators (respectively $(q; \{\mu\})$-oscillators) for which $\varphi(n)$ is polynomial in n (respectively polynomial in $[n]_q$) share the characteristic property: they satisfy the k-bonacci relation, with the coefficients (14) or (37), if the order of polynomial is $r = k-2$. In particular, the Tribonacci relation occurs if the structure function is quadratic. It is worth mentioning a remarkable fact that in the both these cases the coefficients (14) and (37) of the k-bonacci relation *do not depend on the set of numbers* μ_i involved in the polynomial, and depend only on its label (=subscript) and on the number k, linked as $r = k - 2$ to the order of the polynomial. Independence of μ or μ_1, μ_2 is also seen in (42) and (45) for the Pentanacci and Nine-bonacci relations. Instead, the 'initial values' $E_0, E_1, \ldots, E_{k-1}$ for the k-bonacci relation *inevitably depend* on $\mu_1, \mu_2, \ldots, \mu_{k-2}$, as it is given by formula (3) for E_n joint with (5); see also (35), (36).

Moreover, for deformed μ-oscillators whose $\varphi(n)$ is given in (5) we have studied the alternative possibility that the deformed oscillators of this family may also be considered as those obeying the inhomogeneous FR, see (22). However, in this case, unlike the already mentioned property of $\lambda_i^{(k)}$ and $\lambda_i^{(k)}(q)$ in (14) and (37), the coefficients $\tilde{\alpha}_i$, $\tilde{\tilde{\alpha}}_i$ and α_i from (22), (23) *do depend* on the numbers $\mu_i \equiv (\mu_1, \ldots, \mu_2)$ which determine the polynomial structure function (5). This fact is manifest in table 1. Concerning the related class of deformed oscillators, polynomial in $[n]_q$, one can also consider them from the viewpoint of inhomogeneous FR, but with an important change: the sum appearing on the RHS of the analogue of (23) should now be taken over the powers of q^n (instead of the powers of n). We have also shown that deformed oscillators of these two classes can be treated as quasi-Fibonacci ones, i.e. those obeying the relation with 2-term RHS where $\lambda = \lambda(n)$ and $\rho = \rho(n)$.

What we find even more unusual about the situation is the class of deformed oscillators whose $\varphi(n)$ is polynomial in the p, q-bracket $[n]_{p,q}$. Indeed, the structure function quadratic in $[n]_{p,q}$ obeys the Pentanacci relation first while the FR, Tribonacci and Tetranacci relations all fail. Then, $\varphi(n)$ cubic in $[n]_{p,q}$ defines the family of oscillators which are 9-bonacci oscillators, and so on, according to the rule: m-bonacci relation corresponds to the $(k+1)$st order polynomial, in $[n]_{p,q}$, deformation where $m = \frac{1}{2}(k+2)(k+3) - 1$. Finally, let us note that the both $(q; \{\mu\})$- and $(p, q; \{\mu\})$-deformed oscillators can be viewed as quasi-Fibonacci oscillators, in complete analogy with our above treatment (in section 3.2) of $\{\mu\}$-oscillators, and in analogy with the content of our work [17].

Our final remarks concern the issue of physics aspects of the polynomially deformed non-Fibonacci oscillators, in the one-mode (non-covariant) case studied in this paper. We believe these nonlinear oscillators have good potential to find effective applications in a number of quantum-physics branches, from quantum optics [24, 31, 32] to deformed field theory, say, in the spirit of [33, 34]. What concerns special manifestations of just the non-Fibonacci nature of deformed oscillators, is at present we can only mention our recent work [35] on the application of μ-oscillators (being not Fibonacci but quasi-Fibonacci, see [17]) for the constructing respective μ-Bose gas model in analogy with the p, q-Bose gas model treated in [11]. Unlike the latter, for μ-Bose gas the evaluations of (intercepts of) 2-, 3-particle correlations are significantly more involved and do not yield closed expressions: only approximate formulas can be obtained.

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